

September 1996

A Note on Quantum Liouville Theory via Quantum Group

—— An Approach to Strong Coupling Liouville Theory ——

Takashi Suzuki*

Department of Electronics
Hiroshima Institute of Technology
Saeki, Hiroshima 731-51
JAPAN

Abstract

Quantum Liouville theory is annualized in terms of the infinite dimensional representations of $U_q\mathfrak{sl}(2, \mathbb{C})$ with q a root of unity. Making full use of characteristic features of the representations, we show that vertex operators in this Liouville theory are factorized into *classical* vertex operators and those which are constructed from the finite dimensional representations of $U_q\mathfrak{sl}(2, \mathbb{C})$. We further show explicitly that fusion rules in this model also enjoys such a factorization. Upon the conjecture that the Liouville action effectively decouples into the classical Liouville action and that of a quantum theory, correlation functions and transition amplitudes are discussed, especially an intimate relation between our model and geometric quantization of the moduli space of Riemann surfaces is suggested. The most important result is that our Liouville theory is in the strong coupling region, i.e., the central charge c_L satisfies $1 < c_L < 25$. An interpretation of quantum space-time is also given within this formulation.

* e-mail address: stakashi@cc.it-hiroshima.ac.jp

1 Introduction and Setup

Liouville theory, classically, has a deep connection to the geometry of Riemann surfaces. Indeed, the solution to the Liouville equation yields Poincaré metric on the upper-half-plane or the Poincaré unit disk on which Riemann surfaces are uniformized [1]. In the light of this fact it is natural to expect that quantum Liouville theory gives some insights into quantum geometry of surfaces. Physically speaking, this problem is nothing but the quantum gravity of $2D$ space-time. The appearance of the Liouville theory as a theory of quantum gravity, so-called Liouville gravity, was first recognized by Polyakov in the study of non-critical string theory [2]. In the string theory embedded in the D dimensional target space, the partition function is a function $Z[g]$ of the surface metric g , and is invariant under the group of diffeomorphism acting on the metric g , while it transforms covariantly under local rescalings of the surface metric as,

$$Z[e^\Phi \hat{g}] = e^{\frac{C}{48\pi} S_L(\Phi; \hat{g})} Z[\hat{g}]. \quad (1.1)$$

Here $S_L(\Phi : \hat{g})$ is the Liouville action with the background metric \hat{g} , and is written as

$$S_L(\Phi : \hat{g}) = \int_{\Sigma} d^2z \sqrt{\hat{g}} \left\{ \frac{1}{2} \hat{g}^{ab} \partial_a \Phi \partial_b \Phi + \Lambda e^{\Phi(z, \bar{z})} + R_{\hat{g}} \Phi(z, \bar{z}) \right\}, \quad (1.2)$$

where C is the central charge of the total system, the string coordinates and the reparametrization ghosts. The metric on the Riemann surface Σ parameterized by the complex coordinate (z, \bar{z}) is given by $ds^2 = e^{\Phi(z, \bar{z})} \hat{g}_{z\bar{z}} dz d\bar{z}$. We denote by Λ the cosmological constant. $R_{\hat{g}}$ is the Gaussian curvature measured with the background metric \hat{g} and is given by $R_{\hat{g}} = -2\hat{g}^{z\bar{z}} \partial_z \partial_{\bar{z}} \log \sqrt{\hat{g}}$. Up to now, a number of works (there are too many works to cite here, see, e.g., Refs.[3]–[6] and Ref.[7] for a review) have looked at the quantum Liouville gravity and revealed many remarkable results.

One of the important features of the quantum Liouville theory is that it possesses quantum group structure of $U_q \mathfrak{sl}(2, \mathbb{C})$ implicitly. Precisely, vertex operators in the theory can be expressed in terms of the highest weight representations of $U_q \mathfrak{sl}(2, \mathbb{C})$. We should now remember that, as in the classical algebra $\mathfrak{sl}(2, \mathbb{C})$, there are two kinds of highest weight representations of $U_q \mathfrak{sl}(2, \mathbb{C})$, i.e., of finite dimension and of infinite dimension, and they are completely different from each other. Owing to this fact, one can expect that there are two entirely different versions of the quantum Liouville theory. In one version either finite or infinite dimensional representations are well-defined and we will show that which type of representations appears depends on the charges of the vertex operators. The quantum Liouville gravity investigated so far are mainly associated with the finite dimensional representations of $U_q \mathfrak{sl}(2, \mathbb{C})$ [8]. In this case, however, we have a strong restriction on the central charge of the Liouville gravity, $c_L \geq 25$, so-called $D = 1$ barrier. Such gravity is often called the Liouville gravity in the weak coupling regime, or for short, weak coupling Liouville gravity.

On the other hand, one expects that the quantum Liouville gravity associated with the infinite dimensional representations is completely different from the previous one and that it can get rid of the barrier, that is, the central charge may be in the region $1 < c_L < 25$. The Liouville gravity whose central charge is in this region is often called the strong coupling

Liouville gravity. One of the progresses for the strong coupling Liouville gravity has been made in Refs.[9, 10]. They have shown that, upon using infinite dimensional representations of $U_q\text{sl}(2, \mathbb{C})$, consistent gravity theories can be constructed if and only if the central charge takes the special values 7, 13 and 19. This result makes us confident that the Liouville gravity associated with the infinite dimensional representation is in the strong coupling regime. However, as will be explained in Section 3, the deformation parameter q of the algebra $U_q\text{sl}(2, \mathbb{C})$ dealt with in Ref.[10] is not a root of unity. It should be emphasized here that the infinite dimensional representations of $U_q\text{sl}(2, \mathbb{C})$ when q is a root of unity are drastically different from those with generic q [11]. In particular, a new and remarkable feature is that every irreducible infinite dimensional representations necessarily factorizes into a representation of the *classical* algebra $\text{sl}(2, \mathbb{R})$ and a *finite* dimensional one of $U_q\text{sl}(2, \mathbb{C})$. Along this line, it is worthwhile to investigate the Liouville gravity associated with the infinite dimensional representations with q at a root of unity and observe how the characteristic feature of the representations works in the theory of Liouville gravity. Motivated by this, the aim of this paper is to investigate the quantum Liouville gravity via such representations with hoping that such a theory would lead us to a different strong coupling Liouville gravity from [10].

Hereafter, let us use the notations, \mathcal{R}_{Inf}^{root} and \mathcal{R}_{Inf}^{gene} the infinite dimensional representations of $U_q\text{sl}(2, \mathbb{C})$ *with q at a root of unity* and *generic q* , respectively, and \mathcal{R}_{Fin} the finite dimensional ones.

We will see that the fields $\Phi(z, \bar{z})$ of the Liouville theory based on \mathcal{R}_{Inf}^{root} should be expanded around the classical Liouville field $\varphi_{cl}(z, \bar{z})$ as $\Phi = \varphi_{cl} \oplus \phi$, ϕ representing quantum fluctuations around the classical field (see also Ref.[12]). In other words, the metric with which our quantum Liouville theory is constructed should be chosen as $ds^2 = e^{\kappa\phi(z, \bar{z})}\hat{g}_P$ where κ is some constant and \hat{g}_P is the Poincaré metric $\hat{g}_P = e^{\varphi_{cl}(z, \bar{z})}dzd\bar{z}$ playing the role of the background metric. In view of this, it will turn out that our Liouville theory is effectively composed of two Liouville theories, one is the classical Liouville theory $S^{cl}(\varphi_{cl})$ and the other is the quantum Liouville theory $S^q(\phi : \hat{g}_P)$ with respect to the field $\phi(z, \bar{z})$. The sector $S^q(\phi : \hat{g}_P)$ represents quantum fluctuations around $S^{cl}(\varphi_{cl})$ and is the Liouville theory associated with \mathcal{R}_{Fin} . We will explicitly show that our Liouville gravity is actually in the strongly coupled regime, i.e., $1 < c_L < 25$.

The organization of this paper is as follows: Section 2 looks at the classical Liouville theory in some details, especially from the viewpoints of geometric and algebraic structures of the theory. The discussions of the quantum Liouville gravity are given in Section 3. We will review in §3-1 the weak coupling Liouville theory briefly, and other sections, §3.2–§3.5 are devoted to our main concern, i.e., the Liouville theory based on \mathcal{R}_{Inf}^{root} . We will show the decomposition of our theory into the classical theory and the quantum one, and give a concept of *quantum space-time*. We further discuss about fusion rules and transition amplitude. It will turn out that our formulation of the quantum Liouville theory admits a nice interpretation in the context of the geometric quantization of Kähler geometry of moduli space. Here the classical sector, whose appearance is the peculiarity of our formulation, of correlation function plays an important role, i.e., the Hermitian metric of a line bundle over

the moduli space. In section 4, some discussions are given. Appendix reviews the infinite dimensional representations of $U_q\text{sl}(2, \mathbb{C})$ with q at a root of unity.

2 Classical Liouville Theory

In this section, we will introduce some backgrounds of the classical Liouville theory and Riemann surfaces for our later use, especially algebraic and geometric aspects of the theory.

It is known that the classical Liouville theory describes hyperbolic geometry of Riemann surfaces. Let $\Sigma_{g,N}$ be a Riemann surface with genus g and N branch points $\{z_i\}$ of orders $k_i \in \mathbb{N}_{\geq 2}$. The equation $\delta S_L = 0$ yields the Liouville equation of motion

$$\partial\bar{\partial}\Phi(z, \bar{z}) = \frac{\Lambda}{2}e^{\Phi(z, \bar{z})}. \quad (2.1)$$

Here we have made a specific choice of the background metric as $\hat{g}_{ab} = \delta_{z\bar{z}}$. In that case, the Gaussian curvature is given by $R_g = -\Lambda$, i.e., constant curvature. According to the Gauss-Bonnet theorem, $\Sigma_{g,N}$ admits metrics g with constant negative curvature, called the Poincaré metric, for the negative Euler characteristic $\chi(\Sigma_{g,N}) < 0$. Noticing that, for any metric \tilde{g} on $\Sigma_{g,N}$, there exists a scaling factor e^λ such that $g = e^{-\lambda}\tilde{g}$ has constant negative curvature $R_g = -1$, we will set $\Lambda = 1$ hereafter. Note however that such a setting cannot be allowed for the quantum case.

Let us explain the connection between the Liouville theory and the Fuchsian uniformization of Riemann surfaces. The uniformization theorem states that every Riemann surface with negative constant curvature is conformally equivalent to the quotient of the unit disk $D = \{w \in \mathbb{C} | |w| < 1\}$ by the action of a finitely generated Fuchsian group Γ , i.e., $\Sigma_{g,N} \cong D/\Gamma$. In terms of the uniformization map, $J_\Sigma : D \rightarrow \Sigma_{g,N}$ a solution to the Liouville equation (2.1) is written as

$$e^{\varphi_{cl}(z, \bar{z})} = \frac{|\partial_z J_\Sigma^{-1}(z)|^2}{(1 - |J_\Sigma^{-1}(z)|^2)^2}. \quad (2.2)$$

Upon rewriting $J_\Sigma^{-1}(z) = w$, the coordinate on D , the solution (2.2) gives the Poincaré metric on D ,

$$ds^2 = \frac{dw \wedge d\bar{w}}{(1 - w\bar{w})^2}. \quad (2.3)$$

Note that the $PSL(2, \mathbb{R})$ fractional transformations for $J_\Sigma^{-1}(z)$ leave the metric invariant.

The classical Liouville theory is a conformally invariant theory. This is due to the fact that the energy-momentum (EM) tensor defined by $T_{ab} = 2\pi/\sqrt{\hat{g}}(\delta S_L/\delta \hat{g}_{ab})$ is traceless, i.e., $T_{z\bar{z}}^{cl} = 0$. The $(2,0)$ -component is given by

$$T_{zz}^{cl} = \frac{1}{\gamma^2} \left[-\frac{1}{2} \partial_z \varphi_{cl} \partial_z \varphi_{cl} + \partial_z^2 \varphi_{cl} \right], \quad (2.4)$$

and is conserved, i.e., $\partial_{\bar{z}} T_{zz}^{cl} = 0$. Here it should be emphasized that the second term in eq.(2.4) coming from the last term of the action (1.2) is indispensable in order for the EM-tensor to be traceless and satisfy the correct transformation law of a projective connection under holomorphic coordinate change $z = f(\tilde{z})$,

$$\begin{aligned} \tilde{T}_{\tilde{z}\tilde{z}}^{cl}(d\tilde{z})^2 &= \left(\frac{df}{d\tilde{z}}\right)^2 T_{zz}^{cl}(dz)^2 + \frac{1}{\gamma^2} \{f, \tilde{z}\}_S (d\tilde{z})^2, \\ \{f, \tilde{z}\}_S &= \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'}\right)^2. \end{aligned} \quad (2.5)$$

Indeed, using the fact that the metric $e^{\Phi(z, \bar{z})}$ is a primary field of dimension $(1, 1)$, or equivalently, it transforms under the coordinate change as

$$e^{\tilde{\varphi}_{cl}(\tilde{z}, \bar{\tilde{z}})} d\tilde{z} d\bar{\tilde{z}} = \left| \frac{d\tilde{z}}{dz} \right|^2 e^{\varphi_{cl}(z, \bar{z})} dz d\bar{z}, \quad (2.6)$$

one can check that the EM-tensor satisfies the transformation law (2.5). This transformation law of EM-tensor indicates that the central charge c_0 of the classical theory is

$$c_0 = \frac{12}{\gamma^2}. \quad (2.7)$$

We next discuss the algebraic structure of the classical Liouville theory. To see this, let us take the α -th power of the metric (2.2) at a special point P . Geometrically it corresponds to a branch point P on the surface, and α is related to the order k of the branch point as

$$\alpha_i = \frac{1 - k^{-2}}{2\gamma^2}. \quad (2.8)$$

One immediately finds there are two entirely different regions of α , namely, (I) $\alpha = -j \leq 0$, ($j \in \mathbb{Z}/2$) and (II) $\alpha = h > 0$. It lies in this fact that there are two kinds of quantum Liouville gravity, i.e., of strong coupling and weak coupling regimes. In order to see this more explicitly, we calculate the α -th power of eq.(2.2),

$$(I) \quad e^{-j\varphi_{cl}(z, \bar{z})} = \left(\frac{1 - |J_{\Sigma}^{-1}|^2}{\sqrt{\partial_z J_{\Sigma}^{-1} \partial_{\bar{z}} J_{\Sigma}^{-1}}} \right)^{2j} = \sum_{m=-j}^j N_m^j \psi_m^j(z) \psi^{j,m}(\bar{z}) \quad (2.9)$$

$$(II) \quad e^{h\varphi_{cl}(z, \bar{z})} = \left(\frac{\sqrt{\partial_z J_{\Sigma}^{-1} \partial_{\bar{z}} J_{\Sigma}^{-1}}}{1 - |J_{\Sigma}^{-1}|^2} \right)^{2h} = \sum_{r=0}^{\infty} N_r^h \lambda_r^h(z) \lambda^{h,r}(\bar{z}), \quad (2.10)$$

where N_m^j , N_r^h are binomial coefficients. Although we can represent the functions $\psi_m^j(z)$ and $\lambda_k^h(z)$ in terms of a free field, we are not interested in the explicit expressions within the latter discussions. The crucial fact is that $\psi_m^j(z)$ and $\lambda_r^h(z)$ form, respectively, the finite and

the infinite dimensional representations, V_j^{cl} and V_h^{cl} , of $\mathfrak{sl}(2, \mathbb{C})$: Denoting by E_+, E_-, H the generators of $\mathfrak{sl}(2, \mathbb{C})$ satisfying the relations $[E_+, E_-] = 2H, [H, E_\pm] = \pm E_\pm$, these representations are,

$$\begin{aligned} V_j^{cl} &= \{\psi_m^j(z) \mid E_+ \psi_j^j = E_- \psi_{-j}^j = 0, H \psi_m^j = m \psi_m^j, -j \leq m \leq j\}, \\ V_h^{cl} &= \{\lambda_r^h(z) \mid E_- \lambda_0^h = 0, H \lambda_r^h = (h+r) \lambda_r^h, r = 0, 1, \dots\}. \end{aligned} \quad (2.11)$$

One can further show that the chiral sector $\psi_m^j(z)$ and $\lambda_r^h(z)$ satisfy the Poisson-Lie relations of the algebra $\mathfrak{sl}(2, \mathbb{C})$ [15, 16].

$$\{\psi_{m_1}^{j_1}(z_1) \otimes \psi_{m_2}^{j_2}(z_2)\} = -\pi \gamma^2 (r^{j_1 j_2})_{m_1 m_2}^{m'_1 m'_2} \psi_{m'_2}^{j_2}(z_2) \psi_{m'_1}^{j_1}(z_1), \quad (2.12)$$

where $r^{j_1 j_2}$ is the $\mathfrak{sl}(2, \mathbb{C}) \otimes \mathfrak{sl}(2, \mathbb{C})$ valued classical r -matrix, $r = H \otimes H + E_+ \otimes E_-$. The same relations hold also for $\lambda_r^h(z)$. Upon the general philosophy that the quantization of the Poisson-Lie algebra \mathfrak{g} yields the quantum universal enveloping algebra $U_q \mathfrak{g}$ endowed with Yang-Baxter relation, these algebraic structures will be essential later.

As the last comment, it is necessary for the later discussions to summarize the geometrical aspects of the classical Liouville action. Since our main concern will be on the N punctured sphere, i.e., a sphere with N branch points of orders infinity, we will confine ourselves to the surface $\Sigma_{0,N}$ where punctures are located at $\{z_1, z_2, \dots, z_N = \infty\}$. Two Riemann surfaces of this type are isomorphic if and only if they are related by an element of the group $PSL(2, \mathbb{C})$ – a group of all automorphisms of \mathbf{P}^1 . Using this freedom we can normalize such Riemann surfaces by setting $z_{N-2} = 0, z_{N-1} = 1, z_N = \infty$, then $\Sigma_{0,N} = \mathbb{C} \setminus \{z_1, \dots, z_{N-3}, 0, 1\}$. Defining the space of punctures as $\mathcal{T}_N = \{(z_1, \dots, z_{N-3}) \in \mathbb{C}^{\otimes N-3} \mid z_i \neq z_j, \text{ for } i \neq j\}$, one sees that a point in \mathcal{T}_N represents a Riemann surface of the type $\Sigma_{0,N}$. Moreover, if two $\Sigma_{0,N}$'s are connected by an action of the symmetric group $\text{Symm}(N)$, they should be regarded as the same with each other. Hence, we get the moduli space of Riemann surfaces of the type $\Sigma_{0,N}$ as,

$$\mathcal{M}_{0,N} = \mathcal{T}_N / \text{Symm}(N). \quad (2.13)$$

For the surface $\Sigma_{0,N}$, the Liouville field $\varphi_{cl}(z, \bar{z})$ has the following asymptotics near the punctures,

$$\varphi_{cl}(z, \bar{z}) \xrightarrow{z \rightarrow z_i} \begin{cases} -2 \log \epsilon_i - 2 \log |\log \epsilon_i|, & \epsilon_i \equiv |z - z_i|, & \text{for } i \neq N, \\ -2 \log |z| - 2 \log |\log |z||, & & \text{for } i = N. \end{cases} \quad (2.14)$$

Of course, such asymptotics enjoy the Liouville equation (2.1). Denote by \mathfrak{S} a class of fields on $\Sigma_{0,N}$ satisfying asymptotics (2.14). Due to the asymptotics, the action (1.2) diverges for $\varphi_{cl} \in \mathfrak{S}$ and a regularized action has been obtained by Takhtajan and Zograf [18] in a reparametrization invariant manner as

$$\overline{S}_L(\varphi_{cl}) = \lim_{\epsilon \rightarrow 0} \left\{ \int_{\Sigma_\epsilon} d^2 z \left(\partial_z \varphi_{cl} \partial_{\bar{z}} \varphi_{cl} + e^{\varphi_{cl}(z, \bar{z})} \right) + 2\pi N \log \epsilon + 4\pi(N-2) \log |\log \epsilon| \right\}, \quad (2.15)$$

where $\Sigma_\epsilon = \Sigma \setminus \bigcup_{i=1}^{N-1} \{|z - z_i| < \epsilon\} \cup \{|z| > 1/\epsilon\}$. The Euler-Lagrange equation $\delta \overline{S}_L = 0$ derives again the Liouville equation (2.1). Owing to the regularization, however, the classical action $\overline{S}_L(\varphi_{cl})$ is no longer invariant under the action of $\text{Symm}(N)$ on \mathcal{T}_N . According to Ref.[17], one can find, for a real constant k , 1-cocycle f_σ^k of $\text{Symm}(N)$ satisfying

$$\exp\left(\frac{k}{\pi} \overline{S}_L(\varphi_{cl}) \circ \sigma\right) |f_\sigma^k|^2 = \exp\left(\frac{k}{\pi} \overline{S}_L(\varphi_{cl})\right). \quad (2.16)$$

Here $\text{Symm}(N)$ acts on the trivial bundle $\mathcal{T}_N \times \mathbb{C} \rightarrow \mathcal{T}_N$ as $(t, z) \mapsto (\sigma t, f_\sigma^k(t)z)$, where $t \in \mathcal{T}_N$, $z \in \mathbb{C}$ and $\sigma \in \text{Symm}(N)$. It follows that the cocycle f_σ^k defines a Hermitian line bundle $\mathcal{L}_k = \mathcal{T}_N \times \mathbb{C} / \text{Symm}(N)$ over the moduli space $\mathcal{M}_{0,N}$ and the function $\exp(\frac{k}{\pi} \overline{S}_L^{\varphi_{cl}})$ can be interpreted as a Hermitian metric in the line bundle \mathcal{L}_k . This fact will play an important role in the next section.

3 Quantum Liouville Gravity

As we have seen in the previous section, the classical Liouville theory describes the hyperbolic geometry of Riemann surfaces Σ . Indeed, the classical Liouville field $\varphi_{cl}(z, \bar{z})$ is a function on Σ defining the Poincaré metric in terms of the Fuchsian uniformization map. On the contrary, in quantum theory, the Liouville field $\Phi(z, \bar{z})$ expresses quantum fluctuations of the metric and so does the uniformization map $J_\Sigma^{-1}(z)$. In other words, the coordinates (w, \bar{w}) on the unit disk are not ordinary complex numbers but in some sense quantum objects. This is the reason why we regard the quantum Liouville theory as a theory of quantum 2D gravity or quantum geometry of surfaces.

3.1 Introduction of Quantum Liouville Action

We start by summarizing how the quantum Liouville theory appears in the string theory. The original definition of the string partition function is

$$Z = \int d\tau [d\Phi]_g [dX]_g [d(gh)]_g e^{-S_X(X;g) - S_{gh}(b,c;g) - \Lambda_0 \int \sqrt{g}}, \quad (3.1)$$

with τ the Teichmüller parameter, $X = \{X^\mu\}$ the string coordinates embedded in the D dimensional target space and (gh) stands for the ghost coordinates $\{b, c\}$ associated with the diffeomorphism invariance. Choosing a metric slice $g = e^\Phi \hat{g}$ gives the following relation for the path integral measures,

$$[d\Phi]_{e^\Phi \hat{g}} [dX]_{e^\Phi \hat{g}} [d(gh)]_{e^\Phi \hat{g}} = J(\Phi; \hat{g}) [d\Phi]_{\hat{g}} [dX]_{\hat{g}} [d(gh)]_{\hat{g}} \quad (3.2)$$

where $J(\Phi; \hat{g})$ is the Jacobian. The contributions to the Jacobian from $[dX]$ and $[d(gh)]$ was obtained by Polyakov [2], and that from $[d\Phi]$ was postulated [6] so that the partition

function (3.1) finally had the following form,

$$Z = \int d\tau [d\Phi]_{\hat{g}} [dX]_{\hat{g}} [d(gh)]_{\hat{g}} e^{-S_X(X;\hat{g}) - S_{gh}(b,c;\hat{g})} \times e^{-\int d^2\xi \sqrt{\hat{g}} \left(\frac{A}{2} \hat{g}^{ab} \partial_a \Phi \partial_b \Phi + B R_{\hat{g}} \Phi + \Lambda e^{C\Phi} \right)}. \quad (3.3)$$

The constants A, B are determined by calculating the responses of the Weyl rescaling $\hat{g} \rightarrow e^\sigma \hat{g}$ and demanding the invariance of Z , one finds $A = B = \frac{25-D}{48\pi}$. Upon replacing $\Phi \rightarrow \sqrt{\frac{12}{25-D}} \Phi$, we obtain the Liouville action

$$\frac{1}{4\pi} S_L(\Phi : \hat{g}) = \frac{1}{4\pi} \int_{\Sigma} d^2\xi \sqrt{\hat{g}} \left\{ \frac{1}{2} \hat{g}^{ab} \partial_a \Phi \partial_b \Phi + Q_0 R_{\hat{g}} \Phi + \Lambda e^{\gamma\Phi} \right\}, \quad (3.4)$$

where

$$Q_0 = \sqrt{\frac{25-D}{12}}. \quad (3.5)$$

On the other hand, the constant C plays the role of renormalized coupling constant. In eq.(3.4), we have replaced CCC by γ , since the field Φ has been rescaled. and γ will be related to the deformation parameter q of quantum group $U_q \text{sl}(2, \mathbb{C})$.

The energy-momentum tensor of the Liouville sector is obtained as

$$T_{zz} = -\frac{1}{2}(\partial_z \Phi)^2 + Q_0 \partial_z^2 \Phi, \quad T_{z\bar{z}} = 0. \quad (3.6)$$

The second equation is derived upon the equation of motion. The central charge of the quantum Liouville theory is calculated as

$$c_L = 1 + 12Q_0^2. \quad (3.7)$$

Notice that the total conformal anomaly vanishes,

$$c_X + c_L + c_{gh} = D + (26 - D) - 26 = 0, \quad (3.8)$$

which is consistent with the requirement of the conformal invariance.

For the time being, we will concentrate only on the Liouville sector. The correlation function of N vertices are given by

$$\langle V_{\alpha_1}(z_1, \bar{z}_1) V_{\alpha_2}(z_2, \bar{z}_2) \cdots V_{\alpha_N}(z_N, \bar{z}_N) \rangle = \int_{\mathfrak{S}} [d\Phi] e^{-\frac{1}{4\pi} S_L(\Phi; \hat{g})} \prod_{i=1}^N V_{\alpha_i}(z_i, \bar{z}_i), \quad (3.9)$$

where the Liouville vertex operator with charge α_i is given by

$$V_{\alpha_i}(z_i, \bar{z}_i) = e^{\alpha_i \gamma \Phi(z_i, \bar{z}_i)}. \quad (3.10)$$

Notice that, in this definition of the correlation function, the base manifold on which the vertices live is not a manifold with branch points but just the sphere \mathbf{P}^1 . Then the functional

integral is performed over the space \mathfrak{S} of all smooth metrics $\Phi(z, \bar{z})$ on \mathbf{P}^1 . Since puncture corresponds to the branch point of order infinity, all the punctures on $\Sigma_{0,N}$ correspond to the vertex operators with charges $\alpha_i = 1/2\gamma^2$ (see eq.(2.8)). As in the classical theory, (2.9) and (2.10), there are two distinct regions according to the value of the charge α , i.e., (I) $\alpha \leq 0$ and (II) $\alpha > 0$. We will refer these two regions as (I) weak coupling region and (II) strong coupling region.¹ In quantum theory, however, the difference between them becomes more sever than the classical theory. There is a big gap known as the $D = 1$ barrier between two regions. The weak coupling region, studied first by KPZ and DDK, has been investigated by many authors. On the other hand, the quantum Liouville theory in the strong coupling region is a long-standing problem. Our concern in this article is on this region and we will investigate it by using quantum group methods. Before doing this, it is instructive to review the Liouville theory in the weak coupling region.

3.2 Brief Review of the Weak Coupling Region

The standard approach to the weak coupling Liouville theory begins with the action (3.4). Till now remarkable progresses have been done by many authors. Let us, in this section, observe the quantum group aspects of the weak coupling Liouville theory. The quantum group structure appears through the vertex operators, and has been studied extensively in [8, 9]. A crucial fact is that, in this region, the finite dimensional representations of $U_q\mathfrak{sl}(2, \mathbb{C})$ appear. To be precise, writing the vertex operator with charge $-j$ as

$$e^{-j\gamma\Phi(z, \bar{z})} = \sum_{m=-j}^j \mathcal{N}_m^j \Psi_m^j(z) \bar{\Psi}^{j,m}(\bar{z}), \quad (3.11)$$

they have shown that $\Psi_m^j(z)$, $m = -j, \dots, j$ form the $2j + 1$ dimensional representation of $U_q\mathfrak{sl}(2, \mathbb{C})$ with

$$q = e^{\pi i \frac{\gamma^2}{2}}, \quad (3.12)$$

and satisfy the braiding-commutation relations,

$$\Psi_{m_1}^{j_1}(z_1) \otimes \Psi_{m_2}^{j_2}(z_2) = \left(R^{j_1 j_2} \right)_{m_1 m_2}^{m'_1 m'_2} \Psi_{m'_2}^{j_2}(z_2) \otimes \Psi_{m'_1}^{j_1}(z_1), \quad (3.13)$$

where $R^{j_1 j_2}$ is the universal R -matrix of $U_q\mathfrak{sl}(2, \mathbb{C})$. The factorization property of the quantum vertices is a natural guess from the classical result (2.9). The braiding-commutation relation (3.13) reduces to the Poisson-Lie relation (2.12) in the classical limit, $\gamma \rightarrow 0$. This fact is in agreement with the general concept of quantum groups as the quantization of Poisson-Lie algebras of classical Lie groups.

¹According to the standard convention, the Liouville theories which are defined in the spacetime of dimension $D \leq 1$ and $1 < D < 25$ are called, respectively, weak coupling theory and string coupling theory. The reason why we use here the terms weak and strong is that it will be turned out later that vertices defined in the weak (strong) coupling theory carry negative(positive) charges.

An important fact of the region (I) arises in the relation between Q_0 and the renormalized coupling constant γ , that is,

$$Q_0 = \frac{1}{\gamma} + \frac{\gamma}{2}. \quad (3.14)$$

Substituting (3.14) into the general expression of the central charge (3.7), one obtains

$$c_L = 13 + 6 \left(\frac{\gamma^2}{2} + \frac{2}{\gamma^2} \right) \quad (3.15)$$

and finds that the central charge always satisfies $c_L \geq 25$ (≤ 1) for a real (imaginary) γ . Notice that $\gamma\Phi$ should be real since $e^{\gamma\Phi}$ is the metric in this model. The very origin that restricts the theory to the weak coupling region is the relation (3.14). This was first obtained in Ref.[20] by requiring that the EM tensor (3.6) satisfies Virasoro algebra on the cylindrical basis $S^1 \times \mathbb{R}$. Once Virasoro structure is found, one can apply the Coulomb gas formulation of the minimal CFT to the Liouville theory, although the Liouville field is no longer a free field. Let us explain briefly. Notice that the parameter Q_0 corresponds to $i\alpha_0$, α_0 being the background charge in the Coulomb gas formulation and the highest weight vector $\Psi_j^j(z)$ can be related to the primary field of the type $V_{2j+1,1}(z)$ in the Kac's table, where $V_{n,m}(z) =: e^{i\alpha_{n,m}\hat{\phi}(z)}$: with the charge $\alpha_{n,m} = \{(1-n)\alpha_+ + (1-m)\alpha_-\}/2$ and a free field $\hat{\phi}(z)$. Upon comparing eq.(3.14) and the relations $\alpha_0 = (\alpha_+ + \alpha_-)/2$, $\alpha_+\alpha_- = -2$, one immediately sees that the renormalized coupling constant γ is related to one of the screening charges, say, α_+ as $\gamma = i\alpha_+$. With the above connections at hand, the correspondence between the metric $e^{\gamma\Phi(z,\bar{z})}$ and the screening operator $e^{i\alpha_+\hat{\phi}(z)}$ which has conformal dimension 1 is now clear. Thus we obtain the correct dimension (1,1) for the metric $e^{\gamma\Phi(z,\bar{z})}$ [6]. Furthermore, the relation (3.14) is consistent with the quantum group structure of $U_{q\text{sl}}(2, \mathbb{C})$ when the representation is of finite dimension. Indeed, the braiding commutation relation which is deeply related to the q -6j symbols indicates that the operator $e^{-j\gamma\Phi}$ has dimension $-j - \frac{\gamma^2}{2}j(j+1)$. By comparing with the result of CFT that the dimension of the operator is given as $-\frac{1}{2}j\gamma(j\gamma + 2Q_0)$, we then obtain (3.14). Thus in the weak coupling Liouville theory, the relation (3.14) works well and plays an important role, although it is the origin of the severe problem, the $D = 1$ barrier.

3.3 The Strong Coupling Region

Now we are at the stage to enter into the mysterious world of the strong coupling Liouville theory. At the beginning, we propose a Liouville action for the strong coupling region,

$$\hat{S}_L(\Phi : \hat{g}) = S_L + S', \quad \text{with} \quad S' = \lambda \int d^2z \sqrt{\hat{g}} e^{\varphi_{cl}(z,\bar{z})}, \quad (3.16)$$

where S_L is the Liouville action (3.4). We have introduced the additional cosmological term S' whose necessity will be understood later. Note here that, owing to this term, the potential reduces to the classical metric $e^{\varphi_{cl}(z,\bar{z})}$ in the limit $\Phi \rightarrow -\infty$, while, without this term, the

potential vanishes in the limit. We will work, however, not with this action but with the representations of the quantum group $U_{qsl}(2, \mathbb{C})$ in the following discussions.

Let us turn to the quantum group structure. To do this, we introduce vertex operator $e^{h\gamma\Phi(z, \bar{z})}$ denoting the charge h instead of α in this region. By definition the charge h is always positive. As in the weak coupling case, the vertex operator allows holomorphic decomposition as,

$$e^{h\gamma\Phi(z, \bar{z})} = \sum_{r=0}^{\infty} \mathcal{N}_q^{h,r} \Lambda_r^h(z) \bar{\Lambda}^{h,r}(\bar{z}). \quad (3.17)$$

It can be shown that the vectors $\Lambda_r^h(z), r = 0, 1, \dots$ form an infinite dimensional highest weight representation V_h of $U_{qsl}(2, \mathbb{C})$ with (3.12). Thus in this region, infinite dimensional representations appear instead of finite ones in the weak-coupling region. The question here is the relation between Q_0 and the renormalized coupling constant γ . If we require $c_L > 1$ with maintaining the relation (3.14), the constant γ becomes complex. This fact means that, as suggested in the previous section 3.1, the screening charges become complex and, therefore, the dimensions of primaries are not real in general. Gervais and his collaborators [10] have discussed the strong coupling Liouville theory under the situation. Their solution to the problem is summarized in “truncation theorem”, which guarantees that only in the special central charges $c_L = 7, 13$ and 19 , the chiral components of the Liouville vertices form Virasoro Verma modules with real highest weights. Upon the relation (3.14) together with the central charge (3.7), these central charges $c_L = 7, 13$ and 19 give the values of the constants γ , respectively, $\gamma^2 = -1 + i\sqrt{3}$, $2i$ and $\gamma^2 = 1 + i\sqrt{3}$. Notice here that, with the definition eq.(3.12), the parameter q of the quantum group $U_{qsl}(2, \mathbb{C})$ is generic. In such a situation the relation (3.14) is actually consistent. As in the weak coupling region, the braiding commutation relation in the case of infinite dimensional representations shows that the dimension of the operator $e^{h\gamma\Phi}$ is calculated as $h - \frac{\gamma^2}{2}h(h-1)$. By comparing this dimension with that obtained from the CFT method, that is, $-\frac{1}{2}h\gamma(h\gamma - 2Q_0)$, we again obtain (3.14).

Now we would like to ask a question, what will happen if we take the parameter q being a root of unity. With the idea that the infinite dimensional representations \mathcal{R}_{Inf}^{root} and $gncom$ are drastically different from each other, it is quite natural to expect that the quantum Liouville theory associated with \mathcal{R}_{Inf}^{root} has big difference from that based on \mathcal{R}_{Inf}^{gene} . Furthermore, since the q -6j symbols for the representations \mathcal{R}_{Inf}^{root} is completely different from those for \mathcal{R}_{Inf}^{gene} , we could get rid of the relation (3.14) and, therefore, have any other values of the central charge. In the following discussions, the only essential assumption is that our Liouville theory possesses the structure of \mathcal{R}_{Inf}^{root} .

Let q be a p -th root of unity,² i.e.,

$$q = e^{\pi i \frac{p'}{p}} \quad \text{i.e.,} \quad \frac{\gamma^2}{2} = \frac{p'}{p}, \quad (3.18)$$

with $p, p' \in \mathbb{N}$, coprime with each other. Such representations are reported in the Appendix. Once the parameter q is set to the value (3.18), the infinite dimensional highest weight

² In this paper, as a matter of convention, q such that $q^p = -1$ for $\exists p \in \mathbb{N}$ is also called a root of unity.

representations are parameterized by two integers μ and ν such that $h \rightarrow h_{\mu\nu} = \zeta p - j$ where $\zeta = \nu/2p'$, $j = (\mu-1)/2$. We denote by $V_{\mu,\nu}$ the module on the highest weight state of weight $h_{\mu\nu}$ and by $\Lambda_r^{\mu\nu}(z)$ the r -th weight vector in the module $V_{\mu,\nu}$, namely, it corresponds to the r -th holomorphic sector in the decomposition of the vertex operator $e^{h_{\mu\nu}\Phi(z,\bar{z})}$. One immediately sees that the terms $\Lambda_{kp+\tilde{s}}^{\mu\nu}(z)\bar{\Lambda}^{\mu\nu,kp+\tilde{s}}(\bar{z})$, $k = 0, 1, \dots, \tilde{s} = \mu, \dots, p-1$, disappear from the vertex operator $e^{h_{\mu\nu}\Phi(z,\bar{z})}$ because, as shown in eq.(A.6), $\Lambda_{kp+\tilde{s}}^{\mu\nu}$ are the elements in $\mathcal{X}_{\mu,\nu}$, the space of zero norm states, and so the vertex operator is given by means of the summation only of $\Lambda_{kp+s}^{\mu\nu}(z)\bar{\Lambda}^{\mu\nu,kp+s}(\bar{z})$, $s = 0, \dots, \mu-1$, namely, the elements in the irreducible highest weight module $V_{\mu,\nu}^{irr} = V_{\mu,\nu} \setminus \mathcal{X}_{\mu,\nu}$. Therefore we can now use the important theorem in the Appendix, more concretely, the relation (A.18): The irreducible highest weight module $V_{\mu,\nu}^{irr}$ is isomorphic to the tensor product $V_\zeta^{cl} \otimes \mathcal{V}_j$ where V_ζ^{cl} and \mathcal{V}_j are, respectively, the highest weight representation of the classical algebra $\mathfrak{sl}(2, \mathbb{R})$ with highest weight ζ and the $(2j+1)$ -dimensional representation of $U_q\mathfrak{sl}(2, \mathbb{C})$. Owing to the isomorphism, the holomorphic sector factorizes as

$$\Lambda_{kp+s}^{\mu\nu}(z) \mapsto \lambda_k^\zeta(z) \otimes \Psi_m^j(z), \quad (3.19)$$

and also does the anti-holomorphic sector.

Keeping this remarkable facts of \mathcal{R}_{Inf}^{root} in mind, let us look at the N -point correlation function in the Liouville theory associated with \mathcal{R}_{Inf}^{root} ,

$$\begin{aligned} Z^S[\mathbf{m} : \mu_i, \nu_i] &:= \langle V_{\mu_1\nu_1}(z_1, \bar{z}_1) \cdots V_{\mu_N\nu_N}(z_N, \bar{z}_N) \rangle \\ &= \int [d\Phi] e^{-\frac{1}{4\pi}\hat{S}_L(\Phi;\hat{g})} \prod_{i=1}^N V_{\mu_i\nu_i}(z_i, \bar{z}_i), \end{aligned} \quad (3.20)$$

where $V_{\mu_i\nu_i}(z_i, \bar{z}_i) = e^{h_{\mu_i\nu_i}\gamma\Phi(z_i, \bar{z}_i)}$, and $\mathbf{m} = (m, \bar{m})$ stands for the moduli parameters of the surface. We will come back later to the discussions of correlation function and transition amplitude as well. Before that let us enjoy our model with the remarkable nature of \mathcal{R}_{Inf}^{root} . Thanks to the factorizations (3.19) of the holomorphic sectors and the similar factorization of the anti-holomorphic sector, the vertex operators in (3.20) are factorized as,

$$\begin{aligned} V_{\mu_i\nu_i}(z, \bar{z}) &= \sum_{k=0}^{\infty} N_k^{\zeta_i} \lambda_k^{\zeta_i}(z_i) \lambda_{i,k}^{\zeta_i}(\bar{z}_i) \otimes \sum_{m=-j_i}^{j_i} \mathcal{N}_m^{j_i} \Psi_m^{j_i}(z_i) \Psi_m^{j_i,m}(\bar{z}_i) \\ &= e^{\zeta_i\varphi_{cl}(z_i, \bar{z}_i)} \otimes e^{-j_i\tilde{\gamma}\phi(z_i, \bar{z}_i)} \end{aligned} \quad (3.21)$$

where $\varphi_{cl}(z, \bar{z})$ and $\phi(z, \bar{z})$ are, respectively, the classical and the quantum Liouville fields. With the remarkable decompositions of the vertex operators at hand, we can now conjecture that the action also separates effectively as

$$\hat{S}_L(\Phi; \mathbf{1}) \rightarrow S^{cl}(\varphi_{cl}; \mathbf{1}) + S^q(\phi; \hat{g}_P). \quad (3.22)$$

Here the actions $S^{cl}(\varphi_{cl}; \mathbf{1})$ and $S^q(\phi; \hat{g}_P)$ are the Liouville actions with respect to the fields $\varphi_{cl}(z, \bar{z})$ and $\phi(z, \bar{z})$, respectively, and are explicitly given by

$$S^{cl}(\varphi_{cl}; \mathbf{1}) = \frac{1}{\beta^2} \int d^2z \left(\partial_z \varphi_{cl} \partial_{\bar{z}} \varphi_{cl} + \lambda e^{\varphi_{cl}(z, \bar{z})} \right), \quad (3.23)$$

$$S^q(\phi; \hat{g}_P) = \int d^2z \sqrt{\hat{g}_P} \left(\nabla \phi \bar{\nabla} \phi + Q R_{\hat{g}} \phi(z, \bar{z}) + \Lambda e^{\tilde{\gamma} \phi(z, \bar{z})} \right), \quad (3.24)$$

where β is some constant and $\hat{g}_P = e^{\tau \varphi_{cl}} dz d\bar{z}$ is chosen as the background metric for the quantum sector. When $\tau = 1$ in the exponent of \hat{g}_P , the background metric is just the Poincaré metric. The reason why we have introduced new coupling constant β is that, at this stage, we have no idea how the coupling constant of the classical sector should be. On the contrary, remembering that (see Appendix) the finite dimensional representations \mathcal{V}_j appeared by the decomposition has the same deformation parameter q as that of $V_{\mu\nu}^{irr}$, the renormalized coupling constant $\tilde{\gamma}$ of the quantum sector should satisfy,

$$\frac{\tilde{\gamma}^2}{2} = \frac{\gamma^2}{2} + 2n, \quad n = 0, \pm 1, \pm 2, \dots \quad (3.25)$$

It should be emphasized that, upon the condition $\gamma Q_0 = \tilde{\gamma} Q = 1$, the conjecture (3.22) together with (3.23), (3.24) is exactly true under the substitution $\gamma \Phi = \tau \varphi_{cl} + \tilde{\gamma} \phi$ with $\tau = \gamma/\beta$.

Some remarks are now in order. The above observations seems to suggest that, in the quantum Liouville theory via \mathcal{R}_{Inf}^{root} , the Liouville field $\Phi(z, \bar{z})$ should be expanded as $\gamma \Phi = \tau \varphi_{cl} + \tilde{\gamma} \phi$ and, therefore, one can interpret the field ϕ as the quantum fluctuation around $\tau \varphi_{cl}$. In other words, the metric $ds^2 = e^{\gamma \Phi(z, \bar{z})} dz d\bar{z}$ is to be written as $ds^2 = e^{\tilde{\gamma} \phi(z, \bar{z})} \hat{g}_P$ and $e^{\tilde{\gamma} \phi(z, \bar{z})}$ represents quantum fluctuation of metrics around the classical background metric $\hat{g}_P = e^{\tau \varphi_{cl}} dz d\bar{z}$. If τ is set to be 1, the background metric \hat{g}_P becomes the Poincaré metric. Now we can give a possible interpretation of quantum 2D manifold according to the above observations: *Quantum manifold with metric $e^{\gamma \Phi(z, \bar{z})}$ should be considered as the total system of 2D classical manifold with the metric $e^{\tau \varphi_{cl}} dz d\bar{z}$ and quantum fluctuations around the classical surface.* This interpretation matches quite well to the general concept of quantum object. Note also that the action $S^q(\phi)$ governing the quantum fluctuations is again the quantum Liouville theory associated with \mathcal{R}_{Fin} .

It is worthwhile to comment here the difference between our formulation and the standard weak coupling Liouville theory [3]–[6]. In the standard approaches whose action is given in (3.4), the theory also has \mathcal{R}_{Fin} structure as shown in section 3.1. But unlike the quantum sector $S^q(\phi; \hat{g}_P)$ of our model, the background metric can be freely chosen together with the shift of Liouville field $\Phi(z, \bar{z})$. For example, in Refs.[3, 4], the metric in the light-cone gauge $ds^2 = |dx_+ + \mu dx_-|^2$, which is conformally equivalent to the metric in the conformal gauge $ds^2 = e^{\tilde{\gamma} \tilde{\phi}(z, \bar{z})} dz d\bar{z}$ was chosen. These choices have the flat background metric. On the contrary, in our formulation, the background metric is automatically selected as \hat{g}_P . Thus the appearance of the classical sector in addition to the quantum theory associated with \mathcal{R}_{Fin} is the characteristic feature of our formulation. We will recognize the importance of the classical sector from the viewpoints of geometric quantization of moduli space in section 3.5.

3.4 Fusion Rules

Before going to the discussion of the correlation function, it is interesting to look at fusion rules in our model. First, we have to investigate Clebsh-Gordan (CG) decomposition rule for the tensor product of two irreducible infinite dimensional representations of $U_q\mathfrak{sl}(2, \mathbb{C})$,

$$V_1 \otimes V_2 \longrightarrow V_3, \quad (3.26)$$

where $V_i := V_{\mu_i, \nu_i}^{irr} \cong V_{\zeta_i}^{cl} \otimes \mathcal{V}_{j_i}$. The quantum CG coefficient, known as the q -3j symbol, for the infinite dimensional representations of $U_q\mathfrak{sl}(2, \mathbb{C})$ is given when q is *not* a root of unity as follows,

$$\begin{aligned} \left[\begin{array}{ccc} h_1 & h_2 & h_3 \\ r_1 & r_2 & r_3 \end{array} \right]_q^{U_q\mathfrak{sl}(2, \mathbb{C})} &= C(q) \delta_{r_1+r_2, r_3} \tilde{\Delta}(h_1, h_2, h_3) \\ &\times \left\{ \frac{[2j-1][r_3-h_3]![r_1-h_1]![r_2-h_2]![r_1+h_1-1]![r_2+h_2-1]!}{[r_3+h_3-1]!} \right\}^{1/2} \\ &\times \sum_{R \geq 0} (-)^R q^{\frac{R}{2}(r_3+h_3-1)} \frac{1}{[R]![r_3-h_3-R]![r_1-h_1-R]![r_1+h_1-R-1]!} \\ &\cdot \frac{1}{[h_3-h_1-r_1+R]![h_3+h_2-r_1+R-1]!}, \end{aligned} \quad (3.27)$$

where $C(q)$ is a factor which is not important for our analysis below and

$$\begin{aligned} \tilde{\Delta}(h_1, h_2, h_3) &= \{[h_3-h_1-h_2]![h_3-h_1+h_2-1]![h_3+h_1-h_2-1]![h_1+h_2+h_3-2]!\}^{1/2}. \end{aligned} \quad (3.28)$$

The notations $h_i := h_{\mu_i \nu_i} = \zeta_i p - j_i$, $r_i = (\zeta_i + k_i)p - m_i$ have been used. Of course for our case, i.e., $q = \exp(\pi i \frac{p'}{p})$, the CG coefficient is not necessarily well-defined due to the factor $[p] = 0$. What we have to do is only to find conditions which give finite CG coefficients. Since the calculation is lengthy and our interest is not in the details, we describe here only the quite interesting result; finite CG coefficients exist if and only if

$$\begin{aligned} \zeta_1 + \zeta_2 &\leq \zeta_3, \\ |j_1 - j_2| - 1 &< j_3 \leq \min(j_1 + j_2, p - 2 - j_1 - j_2). \end{aligned} \quad (3.29)$$

It should be noticed that, on the modules V^{cl} and \mathcal{V} , the coproducts of the operators $K = q^H$ and L_0 are $\Delta(K) = K \otimes K$ and $\Delta(L_0) = L_0 \otimes 1 + 1 \otimes L_0$, respectively. The coproduct of the Cartan operator yields the conservation law of the highest weights, physically speaking, the conservation of the spin or angular-momentum along the z -axis. Now, from the above coproducts, we have the conservation laws $(\zeta_1 + k_1) + (\zeta_2 + k_2) = (\zeta_3 + k_3)$ and $m_1 + m_2 = m_3 \pmod{p}$. Therefore the minimum value of j_3 is just $|j_1 - j_2|$ because the difference between

$|j_1 - j_2|$ and j_3 is always integer. We therefore obtain the following decomposition rule of the tensor product of two infinite dimensional representations of $U_q\text{sl}(2, \mathbb{C})$ at a root of unity,

$$(V_{\zeta_1}^{cl} \otimes \mathcal{V}_{j_1}) \otimes (V_{\zeta_2}^{cl} \otimes \mathcal{V}_{j_2}) = \left(\bigoplus_{\zeta_1 + \zeta_2 \leq \zeta_3} V_{\zeta_3}^{cl} \right) \otimes \left(\bigoplus_{j_3 = |j_1 - j_2|}^{\min\{j_1 + j_2, p-2-J_1-j_2\}} \mathcal{V}_{j_3} \right). \quad (3.30)$$

The decomposition rules for the tensor products of V_{ζ}^{cl} and of \mathcal{V}_j are the same as those for the tensor products of the $\text{sl}(2, \mathbb{R})$ and of the finite dimensional representation of $U_q\text{sl}(2, \mathbb{C})$, respectively.

Applying this decomposition rule to our model, fusion rule is

$$[e^{h_1\gamma\Phi}] \times [e^{h_2\gamma\Phi}] = \left(\sum_{\zeta_1 + \zeta_2 \leq \zeta_3} [e^{\zeta_3\varphi_{cl}}] \right) \otimes \left(\sum_{j_3 = |j_1 - j_2|}^{\min\{j_1 + j_2, p-2-J_1-j_2\}} [e^{-j_3\tilde{\gamma}\phi}] \right), \quad (3.31)$$

namely, exactly the same as the tensor product the fusion rules of the classical theory and those of the weak coupling theory. This result means that the classical sector and the quantum sector never mix with each other.

3.5 Correlation Functions and Amplitudes

Let us return to the discussion of the correlation functions in our model. The original definition was given in eq.(3.20). What we will see in this section is how the correlation function and amplitude can be written by the decompositions of vertices and actions. Since, as we have seen, the Liouville field Φ can be expanded around the classical solution, the functional measure is $[d\Phi] = [d\phi]$ up to a constant. The correlation function (3.20) is expected finally to be factorized into classical sector and quantum sector, i.e., up to some constant, it can be written as

$$Z^S[\mathbf{m} : \{\mu, \nu\}] = Z^{cl}[\mathbf{m} : \{\zeta\}] Z^q[\mathbf{m} : \{j\}]. \quad (3.32)$$

Here the quantum sector is given by

$$Z^q[\mathbf{m} : \{j\}] := \int [d\phi] e^{-\frac{1}{4\pi} S^q(\phi; \hat{g}_P)} \prod_{i=1}^N e^{-j_i \phi(z_i, \bar{z}_i)}, \quad (3.33)$$

and the classical sector is

$$Z^{cl}[\mathbf{m} : \{\zeta\}] = e^{-\frac{c_0}{48\pi} \bar{S}^{cl}(\varphi_{cl})}, \quad (3.34)$$

where $c_0 = 12/\beta^2$ is the central charge of the classical Liouville theory. The action $\bar{S}^{cl}(\varphi_{cl})$ denotes the classical action defined on the surface with N branch points and regularized by subtracting the singularities near the branch points. In particular, in the case when the topology is the N -punctured sphere, i.e., all the vertices carry the charges $\zeta_i = 1/2\beta^2$, $\bar{S}^{cl}(\varphi_{cl})$ is given by eq.(2.15). Let us discuss in more detail. As stated above, the action

$S^q(\phi; \hat{g}_P)$ corresponds to the Liouville action in the conformal gauge $ds^2 = e^{\tilde{\gamma}\phi(z, \bar{z})}(\hat{g}_P)_{z\bar{z}}dzd\bar{z}$ with \hat{g}_P being the classical background. Suppose that the correlation function $Z^q[m, \{j\}]$ admits holomorphic factorization as,

$$Z^q[m : \{j\}] = \sum_{I,J} N^{I,J} \bar{\Psi}_I[\bar{m} : \{j\}] \Psi_J[m : \{j\}], \quad (3.35)$$

where $N^{I,J}$ is some constant matrix. Here $\Psi_I[m : \{j\}]$ is expected to be a holomorphic section of a line bundle over the moduli space $\mathcal{M}_{0,N}$. We will see later that this expectation is acceptable.

Now we are at the stage to discuss transition amplitudes of our Liouville theory. In order to obtain a transition amplitude, we have to integrate the correlation function $Z[\mathbf{m}]$ over the moduli space of surface metric with the Weil-Peterson metric. Combining (3.34) and (3.35) and integrating over the moduli space, N -point transition amplitude $\mathcal{A}_N(\{\zeta\}, \{j\})$ has the following form,

$$\mathcal{A}_N(\{\zeta\}, \{j\}) = \sum_{I,J} N^{I,J} \int_{\mathcal{M}_{0,N}} d(WP) e^{-\frac{c_0}{48\pi} \bar{S}^{cl}} \bar{\Psi}_I[\bar{m}; \{j\}] \Psi_J[m : \{j\}], \quad (3.36)$$

where $d(WP)$ stands for the Weil-Peterson measure on the moduli space. Here let us see the geometrical meaning of the classical sector. For the topology of the N -punctured sphere, there are quite remarkable facts shown by Zograf and Takhtajan [18] about the connection between the classical Liouville theory and Kähler geometry of the moduli space of complex structure. The important facts are as follows: First the Liouville action evaluated on the classical solution is just the Kähler potential of the Weil-Peterson symplectic structure, precisely, $\omega_{WP} = i\bar{\partial}\partial\bar{S}^{cl}/2$. Second the accessory parameters $c_i, i = 1 \sim N$ are written as $-2\pi c_i = \partial\bar{S}^{cl}/\partial z_i$.³ From these facts, one can show that the functions c_i are in involution [19], i.e., $\{c_i, z_j\}_{WP} = i\delta_{ij}$, where $\{, \}_{WP}$ is the Poisson bracket with respect to the Weil-Peterson symplectic 2-form ω_{WP} . Thus the classical Liouville theory can be regarded as the Kähler geometry of the moduli space of surfaces.

Due to the relation, it is natural to expect the deep connection between our quantum Liouville theory and the geometric quantization of the moduli space which corresponds to the phase space of the classical geometry. With this hope, we should try to observe our result from the viewpoints of geometric quantization. Before doing this, it is helpful to summarize the basic facts about the geometric quantization of a classical theory. Consider a Kähler manifold \mathcal{M} equipped with a symplectic structure ω which is written in terms of the Kähler potential K as $\omega = i\bar{\partial}\partial K$. The Kähler manifold plays the role of the phase space of the classical theory. Geometric quantization is performed by building a line bundle $\mathcal{L} \rightarrow \mathcal{M}$ over the $2N$ dimensional manifold \mathcal{M} with the curvature two form $F = -i\omega$. Let us parameterize \mathcal{M} by (q_i, p_i) , $i = 1 \cdots N$, and denote the section as $\psi(q_i, p_i)$. The final manipulation to complete the geometric quantization is to impose on \mathcal{L} the condition

³This relation between the classical Liouville action and the accessory parameter which is associated with every puncture was first conjectured by Polyakov.[21]

that the section is annihilated by the derivatives of half of the variables, which is called a polarization. In other words, by the choice of a polarization, sections are represented by N variables. Let us, for example, choose the polarization as $\partial_{p_i}\psi = 0$, $i = 0, \dots, N$. Then the Hilbert space \mathcal{H} is the space of sections with the polarization,

$$\mathcal{H} = \{\psi \mid \partial_{p_i}\psi = 0\} \quad (3.37)$$

The inner product on \mathcal{H} is introduced with the Hermitian metric e^{-K} , i.e., for $\psi_1, \psi_2 \in \mathcal{H}$, it is $\langle \psi_1, \psi_2 \rangle = \int e^{-K} \overline{\psi_1} \cdot \psi_2$, where the measure is that defined by ω . Putting our expression (3.36) together with the fact that, as we have seen in the last part of Section 2, a line bundle with the Hermitian metric $\exp(\frac{k}{\pi} \overline{S}^{cl})$ can be constructed on the moduli space $\mathcal{M}_{0,N} = (m_i, c_i)$, the holomorphic part $\Psi_I[m]$ of the quantum sector can be regarded as a holomorphic section of the Hermitian line bundle $\mathcal{L}_{c_0} \rightarrow \mathcal{M}_{0,N}$ with the polarization $\partial_{c_i}\Psi = 0$ and the curvature is $\frac{c_0}{12\pi}\omega_{WP}$. On the other hand, the classical correlation function (3.34) corresponds to a Hermitian metric defining an inner product $\langle \cdot, \cdot \rangle_{c_0}$ on the Hilbert space which is the space of sections $\Psi[m]$. Hence, at least for the topology of the N -punctured sphere, the amplitude can be written as $\mathcal{A} = \sum N^{I,J} \langle \Psi_I, \Psi_J \rangle_{c_0}$.

Thus the quantum Liouville theory associated with \mathcal{R}_{Inf}^{root} fits well with the geometric quantization of moduli space. The factorization property into the classical sector and the quantum sector plays an important role here and is just the special feature appearing only in the Liouville theory associated with \mathcal{R}_{Inf}^{root} .

3.6 Central Charge and Some Discussions

Finally we give some discussions about our Liouville theory. What we have understood in the above discussions is that the quantum Liouville theory based on the infinite dimensional representations of $U_{qsl}(2, \mathbb{C})$ at a root of unity factorizes into the classical Liouville theory and the quantum Liouville theory based on the finite dimensional representations of $U_{qsl}(2, \mathbb{C})$, and the latter governs quantum fluctuations around the classical surface. The following discussions are on this observation, namely, we start from the action

$$\frac{1}{4\pi} \widehat{S}_L = \frac{1}{4\pi\beta^2} \int d^2z \left(\partial\varphi_{cl} \bar{\partial}\varphi_{cl} + e^{\varphi_{cl}} \right) + \frac{1}{4\pi} \int d^2z \sqrt{\hat{g}_P} \left(\nabla\phi \bar{\nabla}\phi + QR_{\hat{g}_P}\phi \right), \quad (3.38)$$

rather than the original one (3.16). Here we have set $\Lambda = 0$ for the convenience. It is important to notice that, since $e^{\gamma\Phi}$ is a Riemannian metric, the Liouville field $\gamma\Phi$ should be real, and, therefore, $\tilde{\gamma}\phi$ is real as well. Recall the relation (3.25) and notice that $\tilde{\gamma}^2$ can be negative. When $\tilde{\gamma}$ is pure imaginary, the quantum field ϕ is also imaginary and the second term in (3.38) has wrong sign, while if $\tilde{\gamma}$ is real, ϕ is a real field and the quantum sector has correct sign. In the following discussions, we would like to impose another assumption that $\beta = \gamma$, equivalently, $\tau = 1$, in order for the background metric \hat{g}_P to be the poincaré metric,

It is easy to calculate EM tensor from the total action, and one finds,

$$T_{zz}^{tot} = T^{cl}(z) + T^q(z) + T^{mix}(z), \quad (3.39)$$

$$\begin{aligned}
T^{cl}(z) &= \frac{1}{\gamma^2} \left(-\frac{1}{2} \partial \varphi_{cl} \partial \varphi_{cl} + \partial^2 \varphi_{cl} \right), \\
T^q(z) &= \lim_{w \rightarrow z} \left[\left(-\frac{1}{2} \partial_z \phi \partial_w \phi + Q \partial^2 \phi \right) - \frac{1}{(z-w)^2} \right], \\
T^{\text{mix}}(z) &= Q \partial \varphi_{cl} \partial \phi, \\
T_{z\bar{z}}^{\text{tot}} &= 0.
\end{aligned} \tag{3.40}$$

The term T^{mix} comes from the curvature term in (3.38) where the classical and quantum Liouville fields φ_{cl} , ϕ interact with each other. The second line (3.40) guarantees that the total system is again conformally invariant.

It is the time to estimate the central charge to confirm that our model is actually a strong coupling theory. To do this, let us observe how the EM tensor $T(z) \equiv T_{zz}^{\text{tot}}$ transforms under the change of variable $z \rightarrow f(z)$. Since the EM tensor is a second rank tensor with central extension, it should satisfy the following transformation law,

$$\hat{T}(z) = \left(\frac{df}{dz} \right)^2 T(f) + \frac{c_L}{12} \{f, z\}_S. \tag{3.41}$$

However, one sees that eq.(3.39) satisfies (3.41) if and only if $Q = 0$, and the central charge is given by

$$c_L = 1 + \frac{12}{\gamma^2}. \tag{3.42}$$

The first term 1 arises from the quantum sector $T^q(z)$, precisely, from the subtraction of the singularity $1/(z-w)^2$,⁴ and the second term $12/\gamma^2$ comes from the classical sector. Putting together the central charge (3.42) with eq.(3.8), one finds

$$\frac{1}{\gamma} = \sqrt{\frac{25-D}{12}} = Q_0. \tag{3.43}$$

Notice that the original coupling constant γ is real for the dimension $D \leq 25$, and that the central charge of our Liouville theory can be actually in the strong coupling region, $1 < c_L < 25$ for $\gamma > 1/\sqrt{2}$. It is interesting to give some of the allowed dimension D of the target space in our model. Recalling the assignment $\gamma^2/2 = p'/p$ with integers p, p' being coprime with each other and (3.43), one finds

$$\begin{aligned}
\text{(i)} \quad & \text{when } p' = 1, \quad D = 1, 7, 13, 19, \\
\text{(ii)} \quad & \text{when } p' = 2, \quad D = 4, 10, 16, 22, \\
\text{(iii)} \quad & \text{when } p' = 3, \quad D = 3, 5, 9, 15, 17, 21, 23. \\
& \dots\dots
\end{aligned} \tag{3.44}$$

⁴ The author thanks N. Anô for a very useful discussion on this

We can obtain fractional dimensions as well. If another matter whose central charge is fractional couples to the string theory, such a fractional dimension will become crucial. In the case (i), the dimensions are completely the same as those given in [8]. This is a strange coincidence, since we are dealing with the case when q is a root of unity, whereas, in [8], the parameter q is not a root of unity.

This central charge (3.42) coincides with that obtained by Takhtajan in Refs.[13, 14], where a manifestly geometrical approach was used. Along the Polyakov's original formulation of quantum Liouville theory, he started with conformal Ward identities via functional integral. Upon perturbation expansions around the classical solution corresponding to the Poincaré metric, he obtained the central charge (3.42). It is worth mentioning that in both approaches, the quantum Liouville theory is expanded around the classical solution, i.e., the Poincaré metric. In other words, a quantum Riemann surface is treated as a total of a classical surface endowed with the hyperbolic geometry and quantum corrections around it. Indeed, in Takhtajan's approach, the classical limit recovers the underlying hyperbolic geometry of Riemann surface. In our case, owing to the condition $Q = 0$, the total action (3.38) is just the sum of the classical Liouville action and a quantum action without any mixing terms. The quantum field $\phi(z, \bar{z})$ can be interpreted as the $D + 1$ -th component of the string, and we obtain the $D + 1$ dimensional spacetime. As suggested previously, the signature the spacetime has depends on the sign of $\tilde{\gamma}^2$.

Next we turn our attention to the conformal dimension. In also this discussion, we are on the conjecture $Q = 0$. The vertex operator of the string dressed by gravity is written as

$$V(z, \bar{z}) = \int d^2z \sqrt{g} : \mathcal{O}_X : \cong \int d^2z e^{\tau \varphi_{cl}} : e^{\tilde{\gamma} \phi} \mathcal{O}_X : \quad (3.45)$$

where \mathcal{O}_X stands for a pure string vertex operator. Let Δ_0, Δ_X be the dimensions of the classical Liouville exponential and the string vertex, respectively, and one finds the following relation to the conformal dimensions

$$\Delta_0 - \frac{\tilde{\gamma}^2}{2} + \Delta_X = 1. \quad (3.46)$$

The middle term in the left hand side of (3.46) is the conformal dimension of the exponential $e^{\tilde{\gamma} \phi}$. Since we have chosen $\tau = 1$, the background metric \hat{g}_P is the Poincaré metric and so $\Delta_0 = 1$. Then, together with (3.25), one finds

$$\frac{\tilde{\gamma}^2}{2} = \Delta_X = \frac{\gamma^2}{2} + 2n, \quad n = 0, \pm 1, \dots \quad (3.47)$$

This equation suggests that the matters (string) which couple to the Liouville theory form a discrete series in conformal dimension and that the constant $\tilde{\gamma}$ can be imaginary.

In summary, our Liouville theory can be interpreted as follows: It is the total system of the classical Liouville theory plus the fluctuation around the classical manifold, which becomes the $D + 1$ -th component of the string. The same situation occurs in the weak coupling Liouville theory, although in that case the classical Liouville sector does not appear. In

the weak coupling theory when the dimension of the target space is $D = 25$, the quantum Liouville field is regarded as the time component of the string with wrong sign and, as the result, the spacetime becomes the 26 dimensional Minkowski space. On the contrary, in our case, we have more choices for the dimension D as listed in (3.44), especially, for $\gamma^2 = \frac{6}{11}$, the dimension of the target space is $D = 3$ and, therefore, with the field ϕ as the time component, our space-time is the 4 dimensional Minkowski or Euclid space according to the sign of the quantum sector S^q .

4 Summary and Discussions

We have developed a new approach to the quantum Liouville theory via the infinite dimensional representations of $U_q\text{sl}(2, \mathbb{C})$ when $q = \exp(\pi i p'/p)$. In this Liouville theory we have dealt with, only the vertex operators with positive charges α can be defined. The characteristic feature of \mathcal{R}_{Inf}^{root} is that every irreducible highest weight module $V_{\mu, \nu}^{irr}$ necessarily factorizes into the highest weight module V_{ζ}^{cl} of the classical algebra $\text{sl}(2, \mathbb{R})$ and \mathcal{V}_j , the $(2j+1)$ -dimensional representation of $U_q\text{sl}(2, \mathbb{C})$. Our investigations in this article have been performed by making full use of this feature. Owing to this fact, we observed that the vertex operators with positive charge $\alpha = h_{\mu\nu}$ factorized into the *classical* vertex operator with charge ζ and the vertex operator with negative charge $-j$ as in eq.(3.21), where the relation $h_{\mu\nu} = \zeta p - j$ was understood. We further conjectured that the Liouville action $\hat{S}_L(\Phi)$ should also decompose into the classical Liouville action $S^{cl}(\varphi_{cl})$ and $S^q(\phi; \hat{g}_P)$ with the classical background $\hat{g}_P = e^{\tau\varphi_{cl}(z, \bar{z})}$, where $e^{\varphi_{cl}}$ defines the Poincaré metric. Since the field $\phi(z, \bar{z})$ can be interpreted as the Liouville field which measures the quantum fluctuations of the metric around the classical metric $e^{\tau\varphi_{cl}}$, the Liouville theory $S^q(\phi; \hat{g}_P)$ governs the theory of quantum fluctuations of quantum Riemann surfaces around the classical surfaces. Namely in our formulation, the quantum Liouville theory $\hat{S}_L(\Phi)$ describes quantum $2D$ space-time as the total system of the classical space-time and the quantum fluctuations around it. We found that the Liouville theory governed by the action $S^q(\phi; \hat{g}_P)$ was associated with the *finite* dimensional representations \mathcal{R}_{Fin} . We remarked the difference between our Liouville theory and the standard Liouville theory [3]–[6] which is confined to the weak coupling region. Although, in both theories, quantum fluctuations of metric are deeply related to \mathcal{R}_{Fin} , our theory contains the classical Liouville theory yielding the underlying Kähler geometry, whereas the latter does not. It is quite important to emphasize that, our quantum Liouville theory is certainly in the strong coupling regime.

Another interesting result was that our Liouville theory fitted well with the concepts of geometric quantization of the moduli space of metrics. First of all, we have noted that the classical Liouville theory is in agreement with the Kähler geometry of the moduli space, where the Liouville action evaluated on the classical solution φ_{cl} is nothing but the Kähler potential of the Weil-Peterson metric ω_{WP} . Second, we have observed that the Hermitian line bundle \mathcal{L}_{c_0} can be built over the moduli space $\mathcal{M}_{0,N}$ with the Hermitian metric $\exp(-\frac{c_0}{48\pi}\bar{S}')$. The role of holomorphic sections of the line bundle \mathcal{L}_{c_0} is played by the holomorphic part $\Psi[m; \{j\}]$

of the quantum sector $Z^q[m : \{j\}]$. Thus, as we have seen in eq.(3.36), the transition amplitude in our theory agrees with the inner product of two wave functions corresponding to the sections of the line bundle \mathcal{L}_{c_0} . Remembering that the quantum Liouville theory can be regarded as a quantum geometry of Riemann surfaces, our observation of our quantum Liouville theory from the viewpoints of geometric quantization of moduli space seems to be quite natural.

At this stage, it is worthwhile to compare our quantum gravity with that formulated in Ref.[22] where $c > 1$ two-dimensional quantum gravity was treated via geometric quantization of moduli space. Riemann surfaces in his consideration are compact and have genus $g > 1$. There, the transition amplitude between some initial state Ψ_I and some final state Ψ_F is given by the inner product (see eq.(5.7) in Ref.[22])

$$\langle \Psi_I, \Psi_F \rangle = \int d(WP) Z_L[m] \bar{\Psi}_I[\bar{m}] \Psi_F[m], \quad (4.1)$$

where $Z_L[m]$ is the Liouville partition function (let $\sigma(z, \bar{z})$ be the Liouville field). $\Psi[m]$ is a holomorphic section of a line bundle over the moduli space and can be identified with the conformal block obtained by solving the conformal Ward identity of Polyakov [3]. Therefore $\Psi[m]$ is considered as the holomorphic sector of the partition function in the weakly coupled Liouville gravity. In that paper, the metric on the Riemann surface is parameterized as $ds^2 = e^{\sigma(z, \bar{z})} |dz + \mu d\bar{z}|^2$. Now it is easy to find intimate relations between the formulation of Verlinde and our Liouville gravity except only one big difference. In both theories, a quantum Riemann surface is composed of two sectors, one is the Riemann surfaces as a background and the other corresponds to quantum fluctuations around the background surface. In the Verlinde's formulation, the quantum fluctuation is parameterized by the Beltrami differential μ . The difference arises in the choice of background surfaces; the background Riemann surface in our theory is just the classical surface with the Poincaré metric $e^{\varphi_{cl}(z, \bar{z})}$, while in the Verlinde's formulation it is again a quantum surface with the metric $e^{\sigma(z, \bar{z})}$. Because $\sigma(z, \bar{z})$ is not a classical field, the partition function $Z_L[m]$ in eq.(4.1) cannot be written as $Z^{cl}[m]$ in our theory. Note again that the holomorphic sections $\Psi[m]$ in both theories are related only to the quantum fluctuations.

Although we have observed some remarkable features of the quantum Liouville theory associated with \mathcal{R}_{Inf}^{root} and obtained a natural concept of quantum $2D$ space-time within the framework of our formulation, there still remain some important problems to be investigated. I list some of them below. First, and maybe most important problem is the explicit relation between our quantum Liouville theory and Takhtajan's one. These two models give the same central charge. Moreover, in both approaches, the classical geometry appears as a background, namely, the quantum metrics are to be expanded around the classical metric of Riemann surface, the Poincaré metric. Inspired by the agreements, it is quite interesting and important to find the relation, although at first glance these two approaches are completely different, i.e., Takhtajan's approach is fully geometric and ours is algebraic. Second, Virasoro structure of this model is an interesting problem. Finding this structure will allow more explicit discussions of our Liouville theory, especially correlation functions. This also maybe

sheds light on the problem of $c > 1$ discrete series of the Virasoro algebra. It is well-known that representations of the Virasoro algebra with $c > 1$ always form continuous series. However, since the highest weight representations \mathcal{R}_{Inf}^{root} form a discrete series and our model can have central charges greater than 1, it can be expected to find such a discrete series. These discussions will appear elsewhere.[23]

Acknowledgment : I wish to thank Profs. L. D. Faddeev and L. Takhtajan for communications on strongly coupled Liouville gravity. It is also a pleasure to acknowledge Profs. M. Ninomiya, R. Sasaki and Dr. N.Ano for discussions and comments.

A Infinite Dimensional Representations with q at a root of unity

The Appendix gives a brief review of Ref.[11] where the infinite dimensional highest weight representations of $U_q\mathfrak{sl}(2, \mathbb{C})$ is examined for the case when the deformation parameter q is a root of unity. Let $q = \exp \pi i/p$.⁵ The most essential feature which we have made full use of in our discussions is stated in the following theorem;

Theorem. Every *infinite* dimensional irreducible highest weight representation, denoted as $V_I^{U_q\mathfrak{sl}(2, \mathbb{C})}$ is necessarily isomorphic to the tensor product of two highest weight representations as

$$V_I^{U_q\mathfrak{sl}(2, \mathbb{C})} \cong V^{\mathfrak{sl}(2, \mathbb{R})} \otimes V_F^{U_q\mathfrak{sl}(2, \mathbb{C})}, \quad (\text{A.1})$$

where $V^{\mathfrak{sl}(2, \mathbb{R})}$ is a representation of the *classical* algebra $\mathfrak{su}(1, 1) \cong \mathfrak{sl}(2, \mathbb{R})$ and $V_F^{U_q\mathfrak{sl}(2, \mathbb{C})}$ is a *finite* dimensional representation of $U_q\mathfrak{sl}(2, \mathbb{C})$.

Below we will prove this theorem (see Refs.[11] for the detailed discussions).

Let $X_+, X_-, K = q^H$ be the generators of the quantum universal enveloping algebra $U_q\mathfrak{sl}(2, \mathbb{C})$ satisfying the relations

$$[X_+, X_-] = \frac{K^2 - K^{-2}}{q - q^{-1}}, \quad KX_{\pm} = q^{\pm 1}X_{\pm}K. \quad (\text{A.2})$$

To get infinite dimensional representations, we define hermitian conjugations as $X_{\pm}^{\dagger} = -X_{\mp}$, $K^{\dagger} = K^{-1}$. As in the classical case, we represent $U_q\mathfrak{sl}(2, \mathbb{C})$ with these conjugations by constructing the highest weight module V_h on the highest weight state Λ_0^h such as $X_- \Lambda_0^h = 0$, $K \Lambda_0^h = q^h \Lambda_0^h$,

$$V_h = \{ \Lambda_r^h \mid \Lambda_r^h := \frac{X_+^r}{[r]!} \Lambda_0^h, \ r = 0, 1, \dots \}. \quad (\text{A.3})$$

⁵In the previous sections, we used the choice $q = \exp \pi i \frac{p'}{p}$. For the brevity of our discussions, we will choose $p' = 1$ in the appendix. The essential parts of our discussion is independent of the choice for p' .

Upon using the hermitian conjugations and the relations (A.2), the norm of the state Λ_r^h is easily calculated as

$$\|\Lambda_r^h\|^2 = \left[\begin{matrix} 2h + r - 1 \\ r \end{matrix} \right]_q \quad (\text{A.4})$$

with the normalization $\|\Lambda_0^h\|^2 = 1$.

The first problem we come across when q is a p -th root of unity is that the norm of the state Λ_p^h diverges owing to the factor $[p] = 0$ in the denominator. Thus, for arbitrary highest weight h , the highest weight module V_h is not necessarily well-defined. The only way to avoid this undesirable situation is to require that there exist two integers $\mu \in \{0, 1, \dots, p\}$ and $\nu \in \mathbb{N}$, such that the highest weight is given by,

$$h = h_{\mu\nu} := \frac{1}{2}(p\nu - \mu + 1). \quad (\text{A.5})$$

For the highest weight $h_{\mu\nu}$, the factor $[2h_{\mu\nu} + \mu - 1] = 0$ appearing in the numerator of the right hand side of the norm (A.4) makes the p -th state a finite-norm state, and so the module $V_{h_{\mu\nu}}$ is well-defined. However, due to the zero, there appears the set of zero-norm states

$$\mathcal{X}_{\mu\nu} = \bigoplus_{k=0}^{\infty} \{\Lambda_{kp+\mu}^h, \Lambda_{kp+\mu+1}^h, \dots, \Lambda_{(k+1)p-1}^h\}. \quad (\text{A.6})$$

One can immediately show that, upon the relation $[kp+x] = (-)^k[x]$, $\|\chi\|^2 = 0$ for $\chi \in \mathcal{X}_{\mu,\nu}$. It is interesting to notice that the $V_{h_{\mu\nu}}$ actually form a discrete series parameterized by the two integers μ and ν . This is a characteristic feature of \mathcal{R}_{Inf}^{root} . Indeed this is not the case for the infinite dimensional representations of the classical algebra $\mathfrak{sl}(2, \mathbb{C})$ and of $U_q\mathfrak{sl}(2, \mathbb{C})$ with generic q as well. It will be turned out that one of the parameter, say, ν parameterizes the classical sector $V^{\mathfrak{sl}(2, \mathbb{R})}$ and the other, μ , is concerned with $V_F^{U_q\mathfrak{sl}(2, \Phi)}$.

Let $V_{\mu,\nu} := V_{h_{\mu\nu}}$ and $\Lambda_r^{\mu\nu} := \Lambda_r^{h_{\mu\nu}}$. The second problem we encounter in the construction of an irreducible highest weight module is that X_{\pm}^p are nilpotent on the module $V_{\mu,\nu}$, i.e., $(X_{\pm})^p \Lambda = 0$ for $\forall \Lambda \in V_{\mu,\nu}$. Therefore one cannot move from a state to another state by acting X_+ or X_- successively. Moreover, the relation $K^{2N} = 1$ on $V_{\mu,\nu}$ indicates that one cannot measure the weight of a state completely. In order to remedy these situations, we should change the definition of $U_q\mathfrak{sl}(2, \mathbb{C})$ by adding new generators,

$$L_1 := -\frac{(-X_-)^p}{[p]!}, \quad L_{-1} := \frac{X_+^p}{[p]!}, \quad L_0 := \frac{1}{2} \left[\begin{matrix} 2H + p - 1 \\ p \end{matrix} \right]_q \quad (\text{A.7})$$

to the original ones, X_+, X_- and $K = q^H$. The complete highest weight module $V_{\mu,\nu}$ is constructed by acting X_+ and L_{-1} on the highest weight state $\Lambda_0^{\mu\nu}$ which is defined by the relations, $X_- \Lambda_0^{\mu\nu} = L_1 \Lambda_0^{\mu\nu} = 0$.

The third problem, which we do not encounter in the classical case and the case when q is not a root of unity, is that there is an infinite chain of submodules in $V_{\mu,\nu}$. That is to say, $V_{\mu,\nu}$ is no longer irreducible. Let us observe this characteristic feature briefly.

First of all, one sees that the state $\Lambda_\mu^{\mu\nu}$ is a highest weight state because both X_- and L_1 annihilate this state. Therefore a submodule exists on the state $\Lambda_\mu^{\mu\nu}$. Since this state has weight $h_{\mu\mu} + \mu = h_{-\mu\nu}$, the submodule can be regarded as $V_{-\mu,\nu}$. Next one again finds a submodule $V_{\mu,\nu+2}$ on $\Lambda_{p-\mu}^{-\mu\nu} \in V_{-\mu,\nu}$. Repeating this procedure, one obtains the following chain of submodules in the original module $V_{\mu,\nu}$,

$$V_{\mu,\nu} \supset V_{-\mu,\nu} \supset V_{\mu,\nu+2} \supset V_{-\mu,\nu+2} \supset \cdots \supset V_{\mu,\nu+2k} \supset V_{-\mu,\nu+2k} \supset \cdots. \quad (\text{A.8})$$

Then, irreducible highest weight module on the highest weight state $\Lambda_0^{\mu\nu}$, denoted by $V_{\mu,\nu}^{irr}$, is obtained by subtracting all the submodules, and it is obtained as

$$V_{\mu,\nu}^{irr} = \bigoplus_{k=0}^{\infty} V_{\mu,\nu}^{(k)}, \quad V_{\mu,\nu}^{(k)} := \{\Lambda_{kp+s}^{\mu\nu} \mid s = 0, 1, \dots, \mu-1\}. \quad (\text{A.9})$$

It should be noticed that all the states subtracted in the procedure are the elements in the set $\mathcal{X}_{\mu,\nu}$ and, therefore, no zero norm state exists in $V_{\mu,\nu}^{irr}$, namely,

$$V_{\mu,\nu}^{irr} = V_{\mu,\nu} \setminus \mathcal{X}_{\mu,\nu}. \quad (\text{A.10})$$

We have now obtained irreducible highest weight modules. Notice that every irreducible infinite dimensional highest weight representation is composed of an infinite number of blocks $V^{(k)}$, $k = 0, 1, \dots$, each of which contains finite number of states. This feature is the very origin of the fact stated in the Theorem (A.1). To see this is the next task.

Now we are the last stage to prove the Theorem. To complete the proof, we make some observations. In the following we will denote the level of a state by $kp + s$ where s runs from 0 to $\mu - 1$ and $k = 0, 1, \dots$, instead of r , $r = 0, 1, \dots$.

OBSERVATION 1. On $V_{\mu,\nu}^{irr}$

$$\frac{X_\pm^{kp+s}}{[kp+s]!} = (-)^{\frac{1}{2}k(k-1)p+ks} \frac{L_{\mp 1}^k}{k!} \frac{X_\pm^r}{[s]!}. \quad (\text{A.11})$$

OBSERVATION 2. The sets of generators $\{X_+, X_-, K\}$ and $\{L_1, L_{-1}, L_0\}$ are mutually commutable on the irreducible highest weight module $V_{\mu,\nu}^{irr}$.

These observations indicate that there exists a map $\rho : V_{\mu,\nu}^{irr} \rightarrow V^{inf} \otimes \mathcal{V}$, where \mathcal{V} is the finite dimensional space composed of μ states. Further ρ induces another map $\hat{\rho} : U_{qsl}(2, \mathbb{C}) \rightarrow U^{inf} \otimes \mathcal{U}$, such that $\rho(\mathcal{O}\Lambda) = \hat{\rho}(\mathcal{O})\rho(\Lambda)$. In the second paper in Ref.[11], such the isomorphisms ρ and $\hat{\rho}$ have been obtained. In the following, we shall restrict the module $V_{\mu,\nu}^{irr}$ to the unitary irreducible representation. In that case,

$$\rho(\Lambda_{kp+s}^{\mu\nu}) = (-)^{\frac{1}{2}k(k-1)p+ks} \lambda_k^\zeta \bigotimes \Psi_m^j, \quad (\text{A.12})$$

and

$$\hat{\rho}(X_\pm) = \mathbf{1} \otimes (\pm \mathcal{J}_\pm), \quad \hat{\rho}(K) = \mathbf{1} \otimes \mathcal{K}, \quad (\text{A.13})$$

$$\hat{\rho}(L_{\pm 1}) = (\mp G_{\pm 1}) \otimes \mathbf{1}, \quad \hat{\rho}(L_0) = G_0 \otimes \mathbf{1}, \quad (\text{A.14})$$

where $\zeta = \nu/2$, $j = (\mu - 1)/2$ and $m = -j + s$.

OBSERVATION 3. The actions of $\{\mathcal{J}_\pm, \mathcal{K}\} \in \mathcal{U}$ on the state $\Psi_m^j \in \mathcal{V}$ are calculated as

$$\mathcal{J}_\pm \Psi_m^j = [j \pm m + 1] \Psi_{m\pm 1}^j, \quad \mathcal{K} \Psi_m^j = q^m \Psi_m^j, \quad (\text{A.15})$$

OBSERVATION 4. The actions of $G_n, (n = \pm 1, 0) \in U^{inf}$ on $\lambda_k^\zeta \in V^{inf}$ are as follows,

$$G_1 \lambda_k^\zeta = (2\zeta + k - 1) \lambda_{k-1}^\zeta, \quad G_{-1} \lambda_k^\zeta = (k + 1) \lambda_{k+1}^\zeta, \quad G_0 \lambda_k^\zeta = \left(\frac{1}{2} \zeta + k \right) \lambda_k^\zeta. \quad (\text{A.16})$$

The OBSERVATION 3 shows that $\mathcal{J}_\pm, \mathcal{K}$ satisfy the relations of $U_{q\text{sl}}(2, \mathbb{C})$, i.e., $\mathcal{U} = U_{q\text{sl}}(2, \mathbb{C})$ with the hermitian conjugations $\mathcal{J}_\pm^\dagger = \mathcal{J}_\mp, \mathcal{K}^\dagger = \mathcal{K}^{-1}$. Therefore, taking it into account that Ψ_m^j has positive norm, \mathcal{V} is a unitary finite dimensional representation of $U_{q\text{sl}}(2, \mathbb{C})$ with the highest weight j . We rewrite such the representation as \mathcal{V}_j , i.e., $\dim \mathcal{V}_j = 2j + 1$. On the other hand, OBSERVATION 4 leads us to the following relations among $G_n, n = 0, \pm 1$,

$$[G_n, G_m] = (n - m) G_{n+m}, \quad (\text{A.17})$$

and hermitian conjugations $G_{\pm 1}^\dagger = G_{\mp 1}, G_0^\dagger = G_0$. Therefore we can conclude that U^{inf} is just the classical universal enveloping algebra of $\mathfrak{su}(1, 1) \cong \mathfrak{sl}(2, \mathbb{R})$, i.e., $U^{inf} = U\mathfrak{sl}(2, \mathbb{R})$ and V^{inf} is the unitary highest weight representation of $\mathfrak{sl}(2, \mathbb{R})$ with highest weight ζ . We denote the representation by V_ζ^{cl} . We have now finished the proof of the theorem and obtained the important structure of \mathcal{R}_{Inf}^{root} as

$$V_{\mu, \nu}^{irr} \cong V_\zeta^{cl} \otimes \mathcal{V}_j. \quad (\text{A.18})$$

In the Theorem, $V_{\mu, \nu}^{irr}, V_\zeta^{cl}$ and \mathcal{V}_j are denoted by $V_I^{U_{q\text{sl}}(2, \mathbb{C})}, V^{\mathfrak{sl}(2, \mathbb{R})}$ and $V_F^{U_{q\text{sl}}(2, \mathbb{C})}$, respectively.

References

- [1] H. Poincaré. J. Math. Pure **5 se. 4** (1898) 365.
- [2] A. Polyakov. Phys. Lett. **103B** 207, 211.
- [3] A. Polyakov. Mod. Phys. Lett. **A2** (1987) 893.
- [4] V. Knizhnik, A. Polyakov, A. B. Zamolodchikov. Mod. Phys. Lett. **A3** (1988) 819.
- [5] F. David. Mod. Phys. Lett. **A3** (1988) 1651.
- [6] J. Distler, H. Kawai. Nucl. Phys. **B321** (1989) 509.
- [7] See, e.g., N. Seiberg. Prog. Theor. Phys. (Suppl.) **102** (1990) 319.
P. Ginsparg, G. Moore. *Lectures on 2D Gravity and 2D String Theory*, Lectures given at TASI summer school, 1992, hep-th/9304011. and references therein.

- [8] J.-L. Gervais. Phys. Lett. **243B** (1990) 85, Commun. Math. Phys. **130** (1990) 257, Commun. Math. Phys. **138** (1991) 301.
J.-L. Gervais, B. Rostand. Commun. Math. Phys. **143** (1991) 175.
E. Cremmer, J.-L. Gervais. Commun. Math. Phys. **144** (1992) 279.
- [9] F. Smirnov, L. Takhtajan. *Towards a Quantum Liouville Theory with $c > 1$* , Univ. of Colorado preprint, 1990.
- [10] J.-L. Gervais, A. Neveu. Phys. Lett. **151B** (1985) 271.
J.-L. Gervais, B. Rostand. Nucl. Phys. **B346** (1990) 473.
J.-L. Gervais. Commun. Math. Phys. **138** (1991) 301.
J.-L. Gervais, J.-L. Roussel. Nucl. Phys. **426** (1994) 140.
J.-L. Gervais, J. Schnittger. Phys. Lett. **315B** (1993) 258, Nucl. Phys. **B431** (1994) 273.
- [11] T. Matsuzaki, T. Suzuki. J. Phys. **A26** (1993) 4355.
T. Suzuki. J. Math. Phys. **35** (1994) 6857.
- [12] T. Suzuki. Phys. Lett. **B338** (1994) 175.
- [13] L. Takhtajan. Mod. Phys. Lett. **A8** (1993) 3529; **A9** (1994) 2293.
- [14] L. Takhtajan. *Topics of Quantum Geometry of Riemann Surfaces: Two-Dimensional Quantum Gravity*, hep-th/9409088.
- [15] J. -L. Gervais, A. Neveu. Nucl. Phys. **B199** (1982) 59, **209** (1982) 125.
- [16] L. D. Faddeev. L. Takhtajan. *in* Lecture Note in Physics **246** (1986) 166.
- [17] P. Zograf. Leningrad Math. J. **1** (1990) 941.
- [18] P. Zograf, L. Takhtajan. Funct. Anal. Appl. **19** (1986) 216, Math. USSR Sbornik **60** (1988) 143.
- [19] L. Takhtajan. *Uniformization, local index theorem, and the geometry of Riemann surfaces and vector bundles*, Proc. of Symposia in Pure Math. **49** (1989) 581.
- [20] T. Curtright, C. Thorn. Phys. Rev. Lett. **48** (1982) 1309,
E. Braaten, T. Curtright, C. Thorn. Phys. Lett. **118B** (1982) 115,
E. Braaten, T. Curtright, G. Ghandour, C. Thorn. Phys. Rev. Lett. **51** (1983) 19; Ann. Phys. **147** (1984) 365.
- [21] A. Polyakov. Lectures at Steklov, Leningrad 1982, *unpublished*.
- [22] H. Verlinde. Nucl. Phys. **B337** (1990) 652,
- [23] N. An, T. Suzuki. in progress.